Structure of Compact Quantum Groups $A_u(Q)$ and $B_u(Q)$ and their Isomorphism Classification

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1. The Notion of Quantum Groups

- $G =$ Simple compact Lie group, e.g.

  $$G = SU(2) = \left\{ \begin{bmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{bmatrix} : \alpha\bar{\alpha} + \gamma\bar{\gamma} = 1, \alpha, \gamma \in \mathbb{C} \right\}.$$

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$$\Delta : A \to A \otimes A, \quad \Delta(f)(s,t) = f(st).$$

$$\varepsilon : A \to \mathbb{C}, \quad \varepsilon(f) = f(e).$$

$$S : A \to A, \quad S(f)(t) = f(t^{-1}).$$

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Idea of Quantization:

Commuting functions on G, e.g. α, γ,

$$\Downarrow$$

Non-commuting operators, e.g. α, γ,

Commutative $C(G) \Longrightarrow$ Noncommutative $C(G_q)$.

G_q = quantum group

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  Characterization of Lie groups among topological groups.

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- Lesson:

  Quantums groups $=$ “nice Hopf algebras”

  Restrict to such Hopf algebras to obtain nice and deep theory.

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  (2) $\exists\, u^{-1}$ in $M_n(A)$ and anti-morphism S on $\mathcal{A} = * - alg(u_{ij})$ with

$$S(S(a^*)^*) = a, \quad a \in \mathcal{A}; \quad S(u) = u^{-1}.$$

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  (2′) $\exists\ u^{-1}$ and $(u^t)^{-1}$ in $M_n(A)$

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  (2′) $\exists\, u^{-1}$ and $(u^t)^{-1}$ in $M_n(A)$
- There are other equivalent definitions of CMQG

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Relations: the 2×2 matrix $u := \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$ is unitary, Famous Example: $C(SU_q(2))$, $q \in \mathbb{R}$, $q \neq 0$

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i.e.

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,$$

$$\gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

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- Hopf algebra structure same as $C(SU(2))$:

$$\begin{aligned} \Delta(u_{ij}) &= \sum_{k=1}^{2} u_{ik} \otimes u_{kj}, \qquad i,j = 1,2; \\ \varepsilon(u_{ij}) &= \delta_{ij}, \qquad i,j = 1,2; \\ S(u) &= u^{-1}. \end{aligned}$$

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- Yes: Universal quantum groups $A_u(Q)$ and $B_u(Q)$, quantum permutation groups $A_{aut}(X_n)$, etc.

2. Universal CMQGs $A_u(Q)$ and $B_u(Q)$

For $u = (u_{ij})$, $\bar{u} := (u_{ij}^*)$, $u^* := \bar{u}^t$; Q is an $n \times n$ non-singular complex scalar matrix.

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Every CMQG with self conjugate fundamental representation is a quantum subgroup of $B_u(Q)$ for some Q.

- The C^*-algebras $A_u(Q)$ and $B_u(Q)$ are non-nuclear (even non-exact) for generic Q's.

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2. $A_u(Q)$ and $B_u(Q)$ (cont.)

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- Banica's computed the fusion rings of $A_u(Q)$ (for $Q > 0$) and $B_u(Q)$ (for $Q\bar{Q}$) in his deep thesis.

3. $A_u(Q)$ for Positive Q

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THEOREM 2. Let $Q \in GL(n, \mathbb{C})$ and $Q' \in GL(n', \mathbb{C})$ be positive, normalized, with eigen values $q_1 \geq q_2 \geq \cdots \geq q_n$ and $q'_1 \geq q'_2 \geq \cdots \geq q'_{n'}$ respectively.

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(ii) $(q_1, q_2, \cdots, q_n) = (q'_1, q'_2, \cdots, q'_n)$ or

$$(q_n^{-1}, q_{n-1}^{-1}, \cdots, q_1^{-1}) = (q'_1, q'_2, \cdots, q'_n).$$

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Note: The quantum groups $B_u(Q)$ are *simple* in an appropriate sense: cf. S. Wang: “Simple compact quantum groups I”, JFA, 256 (2009), 3313-3341.

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$$A_u(Q) \cong A_u(P_1) * A_u(P_2) * \cdots * A_u(P_k).$$

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Note: This is contrary to an earlier belief that $A_u(P_i)$'s do not appear in the decomp.!

COROLLARY of THEOREM 4.

(1). Let $Q = diag(e^{i\theta_1}P_1, e^{i\theta_2}P_2, \cdots, e^{i\theta_k}P_k)$, with positive matrices P_j and distinct angles $0 \leq \theta_j < 2\pi$, $j = 1, \cdots, k$, $k \geq 1$

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(2). Let $Q \in GL(2, \mathbb{C})$ be a non-normal matrix. Then $A_u(Q) = C(\mathbb{T})$.

(3). For $Q \in GL(2, \mathbb{C})$, $A_u(Q)$ is isomorphic to either $C(\mathbb{T})$, or $C(\mathbb{T}) * C(\mathbb{T})$, or $A_u(diag(1, q))$ with $0 < q \leq 1$.

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(1). Let $Q = diag(T_1, T_2, \cdots, T_k)$ be such that $T_j \bar{T}_j = \lambda_j I_{n_j}$, where the λ_j's are distinct non-zero real numbers (the n_j's need not be different), $j = 1, \cdots, k$, $k \geq 1$.

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(2). Let $Q = \begin{bmatrix} 0 & T \\ q\bar{T}^{-1} & 0 \end{bmatrix}$, where $T \in GL(n, \mathbb{C})$ and q is a complex but non-real number. Then $B_u(Q)$ is isomorphic to $A_u(|T|^2)$.

6. Important Related Work

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Debashish Goswami used $A_u(Q)$ to prove the existence of quantum isometry groups for very general classes of noncommutative spaces in the sense of Connes.

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$$\pi_x \otimes \pi_y = \sum_{x=ag, \bar{g}b=y} \pi_{ab}$$

Banica's Fusion Theorem 2:

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The irreducible representations π_k of the quantum group $B_u(Q)$ are parameterized by $k \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$, and one has the fusion rules

$$\pi_k \otimes \pi_l = \pi_{|k-l|} \oplus \pi_{|k-l|+2} \oplus \cdots \oplus \pi_{k+l-2} \oplus \pi_{k+l}$$

7. Open Problems

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- (2) Construct explicit models of irreducible representations π_k $(k \in \mathbb{Z}_+)$ of the quantum groups $B_u(Q)$ for Q with $Q\bar{Q} = \pm 1$.

THANKS FOR YOUR ATTENTION!